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Artificial Compressibility Methods for Numerical Solutions of Transonic Full Potential Equation

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New methods for transonic flow computations based on the full potential equation in conservation form are presented. The idea is to modify slightly the density (due to the artifical viscosity in the supersonic region), and solve the resulting elliptic-like problem iteratively. It is shown that standard discretization techniques (central differencing) as well as some standard iterative procedures (SOR, ADI, and explicit methods) are applicable to the modified transonic mixed-type equation. Calculations of transonic flows around cylinders and airfoils are discussed with special emphasis on the explict methods that are suitable for vector processing on the STAR 100 computer.

I. Introduction

FOR many aerodynamic calculations, the assumption of potential flow is meaningful and useful. In 1970, Murman and Cole[1] introduced type-dependent, finite-difference relaxation methods for the solution of a transonic small-disturbance equation. The full potential, rather than the small perturbation one, is usually required for real configurations. Jameson extended Murman's scheme[2] and developed highly sophisticated numerical methods to solve the full-potential equation in the transonic range with remarkable success (e.g., rotated difference schemes,[3] fully conservative schemes,[4] and, recently, finite volumes[5]).

In Jameson's work, an artificial viscosity term in the supersonic region is added to the governing equation implicitly via upwind differencing as in the rotated difference schemes, or explicitly as in the conservative schemes. The second feature is that the iterative procedure, based on line relaxation, is constructed such that in the supersonic region the artificial time-dependent equation describing the development of the solution through iteration is essentially hyperbolic with the flow direction[3] as the time-like direction (with the addition of the artificial viscosity, this equation becomes parabolic). Based on Von Neumann stability analysis, Jameson concluded that no damping term in the artificial time t (namely ϕ_t) is allowed in the supersonic region; hence, the relaxation factor is kept at one in the supersonic region and greater than one (and less than two) in the subsonic region. For stability augmentation, in particular near the sonic line, extra time-dependent terms (ϕ_{st}) are sometimes needed and are added to the scheme.

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Due to complications in implementing Jameson's schemes, special skills are needed for transonic full-potential calculations, especially for general mesh and complex geometries.

In this paper we present a new, simple, and straightforward method to solve the full potential equation in conservation form. With a slight modification of the density in the supersonic region (within the same order of the truncation error), the transonic flow problem becomes amenable to computations by standard discretization techniques and some standard iterative procedures. The density modification is, in fact, equivalent to an artificial viscosity. The magnitude and the form of the modified density and hence the artificial viscosity term are critical to the success of the artificial compressibility methods. The additional artificial term in the expression for the modified density is discretized with a bias in the upwind direction. Except for such modifications, the transonic mixed-type equation is treated as if it were elliptic with the modified density as a given function of the independent variables. For example, unlike Jameson's scheme, standard line overrelaxation method is applicable without special treatment of the supersonic points.

Based on some numerical experiments, the method seems to be satisfactory and the results are encouraging. The purpose of this paper is to report on some of these numerical experiments. In Sec. II, the full-potential equation and the artificial compressibility method are discussed, including the relevant work of Harten[6] and Eberle.[7] The discrete problem is described in Sec. III. The artificial time-dependent equation describing the standard line overrelaxation procedure applied to the modified density equation is discussed in Sec. IV and compared to Jameson's schemes. Other iterative procedures, including ADI and explicit methods, are also discussed and some numerical results are presented. Section V is about a feasibility study of vector processing of the explicit method on the STAR 100 computer.

II. Artificial Compressibility Method

Starting from Navier-Stokes equations for compressible flows in conservation form,[8] namely, continuity, momentum, and energy equations, we have

$$\frac{\partial(\rho u_j)}{\partial x_j}=0 \tag{1}$$

$$\frac{\partial \rho u_i u_j}{\partial x_j}=-\frac{\partial P}{\partial x_i}+\frac{\partial \tau_{ik}}{\partial x_k} \tag{2}$$

$$\frac{\partial J\rho u_j}{\partial x_j}=\frac{\partial u_j \tau_{jk}}{\partial x_k} \tag{3}$$

where τ_{jk} is the stress tensor and

$$J=\tfrac{1}{2}q^2+h=\tfrac{1}{2}q^2+[a^2/(\gamma-1)] \tag{4}$$

For inviscid isentropic (hence irrotational) flows, the preceding system reduces to the full potential equation (written for two-dimensional flows)

$$(\rho\phi_x)_x+(\rho\phi_y)_y=0 \tag{5}$$

where

$$\rho=(M_\infty^2 a^2)^{1/(\gamma-1)}=\left[1-\frac{\gamma-1}{2}M_\infty^2(\phi_x^2+\phi_y^2-1)\right]^{1/(\gamma-1)} \tag{6}$$

Equation (5) admits compression as well as expansion shocks. In order to capture only the physically acceptable compression shocks, viscous effects are retained. There are two obvious ways to do that:

1) Assume J is constant, hence ρ is given in terms of the velocity through the isentropic relation, Eq. (6), and the energy equation, Eq. (3), becomes:

$$J\left(\frac{\partial\rho\phi_x}{\partial x}+\frac{\partial\rho\phi_y}{\partial y}\right)=\frac{\partial u_j\tau_{jk}}{\partial x_k} \tag{7a}$$

If we rewrite this as

$$\frac{\partial\rho\phi_x}{\partial x}+\frac{\partial\rho\phi_y}{\partial y}=\frac{1}{J}\frac{\partial u_j\tau_{jk}}{\partial x_k}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y} \tag{7b}$$

we recover Jameson's artificial viscosity form. Thus, we can interpret Jameson's method as solving the energy equation.

2) If we insist on satisfying the continuity equation, Eq. (1), then the energy equation, Eq. (3), becomes

$$\rho\phi_x J_x+\rho\phi_y J_y=\frac{\partial u_j\tau_{jk}}{\partial x_k} \tag{8}$$

that is,

$$\rho\frac{\partial}{\partial s}J=\frac{\partial u_j\tau_{jk}}{\partial x_k}\approx-\rho\frac{\partial}{\partial s}J'$$

where

$$\frac{\partial}{\partial s}=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

Equation (8) may be simplified to the form

$$\rho(\partial/\partial s)(J+J')=0 \tag{9}$$

where J' is due to the viscous terms. Equation (9) suggests writing the density in the form

$$\tilde\rho=\{1-[(\gamma-1)/2]M_\infty^2(\phi_x^2+\phi_y^2-\epsilon(uf_x+vg_y)-1)\}^{1/(\gamma-1)} \tag{10}$$

where the term $\epsilon(uf_x+vg_y)$ is the modification due to the viscous effects and the continuity equation reads:

$$(\tilde\rho\phi_x)_x+(\tilde\rho\phi_y)_y=0 \tag{11}$$

The artificial compressibility method is based on Eq. (11) where the density is modified according to an artificial viscosity term, not necessarily the physical one given in Eqs. (7) and (8). The form and the magnitude of the viscous terms are determined by numerical stability requirements.

The idea of artificial compressibility is not completely new. Harten[6] showed for a single conservation law described by a first-order hyperbolic equation $(u_t+f_x(u)=0)$ that the narrowness of shock transition is due to global convexity of the flux function $f(u)$. He modified standard finite-difference schemes by adding an artificial flux term which has this property, and claimed that the artificial compressibility method is superior to the artificial viscosity (Lax-Wendroff), in particular for capturing contact discontinuity.

Artificial Viscosity Forms

The most reliable scheme for transonic full-potential calculation is perhaps Jameson's rotated differences. Rearranging terms, Eq. (5) can be written in the form

$$-(\rho/a^2)[(a^2-q^2)\phi_{ss}+a^2\phi_{nn}]=0 \tag{12}$$

where

$$\phi_{ss}=\frac{u^2}{q^2}\phi_{xx}+\frac{2uv\phi_{xy}}{q^2}+\frac{v^2}{q^2}\phi_{yy}$$

$$\phi_{nn}=\frac{v^2}{q^2}\phi_{xx}-\frac{2uv\phi_{xy}}{q^2}+\frac{u^2}{q^2}\phi_{yy}$$

For supersonic points, an artificial viscosity term is effectively introduced by upwind differencing the terms contributing to ϕ_{ss} and it can be considered as an approximation to the following expression:

$$[1-(a^2/q^2)][\Delta x(u^2u_{xx}+uvv_{xx})+\Delta y(uvu_{yy}+v^2v_{yy})] \tag{13}$$

In Jameson's fully conservative calculations, an equivalent expression is used, namely,

$$-(\mu|u|\Delta x\rho_x)_x-(\mu|v|\Delta y\rho_y)_y \tag{14}$$

where

$$\mu=\max\{0,\ [1-(a^2/q^2)]\}$$

Equations (13) and (14) have the same terms containing the highest derivatives of ϕ.

Jameson's fully conservative scheme can be written in an equivalent artificial compressibility form where

$$\tilde\rho=\rho+\mu(\rho/a^2)(uu_x\Delta x+vv_y\Delta y) \tag{15}$$

The artificial viscosity associated with the artificial compressibility given by Eq. (15) is equivalent to Eq. (14) provided $u_y=v_x$.

Artificial viscosity terms may not be necessarily related to upwind differencing, but rather as an approximation to the viscous terms in Navier-Stokes Eqs. (7) and (8). Viscous transonic small-disturbance equation is derived by Sichel[9] and is given by:

$$(1-M^2)\phi_{xx}+\phi_{yy}=-\epsilon\phi_{xxx} \tag{16}$$

where ϵ is a positive constant.

Equation (16) is parabolic[9] and the problem is well posed. For full-potential calculations, and analog of Eq. (16) is:

$$(1-M^2)\phi_{ss}+\phi_{nn}=-\epsilon\phi_{sss} \tag{17}$$

The leading term in the artificial viscosity expression is:

$$\frac{u}{q}\left(\frac{u^2}{q^2}u_{xx}+\frac{3uv}{q^2}v_{xx}\right)+\frac{v}{q}\left(\frac{3uv}{q}u_{yy}+\frac{v^2}{q^2}v_{yy}\right) \tag{18}$$

An equivalent expression is given by:

$$\rho_{ss} = -\frac{\rho}{a^2} q q_s = \frac{u}{q}\left(\frac{u}{q}\rho_x + \frac{v}{q}\rho_y\right)_x + \frac{v}{q}\left(\frac{u}{q}\rho_x + \frac{v}{q}\rho_y\right)_y \quad (19)$$

or in a conservation form

$$(u\rho_s)_x + (v\rho_s)_y \quad (20)$$

In our calculations, an artificial viscosity similar to Eq. (20) that vanishes in the subsonic region is used, namely,

$$-(\mu u \rho_s \Delta s)_x - (\mu v \rho_s \Delta s)_y \quad (21)$$

where

$$\rho_s \Delta s \simeq \frac{u}{q}\rho_x \Delta x + \frac{v}{q}\rho_y \Delta y \quad (22)$$

Hence, the modified density is:

$$\tilde{\rho} = \rho - \mu \rho_s \Delta s \quad (23)$$

Recently, Eberle[7] used a similar expression in a finite-element calculation of full-potential equation. His expression reads

$$\tilde{\rho} = \rho - \mu \Delta \rho \quad (24)$$

where

$$\Delta\rho = \rho - \rho_H$$

and ρ_H is the value of ρ at an auxiliary point H. It is not clear to the authors, from Eberle's report,[7] however, how the auxiliary point H is chosen. In fact, Eq. (22) is equivalent to Eberle's formula where the auxiliary point is systematically determined. In implementing Eq. (22) we used upwind differencing to approximate ρ_x and ρ_y.

Another variant of the artificial compressibility method is based on modifying the speed of sound, namely,

$$\tilde{a}^2 = 1/M_\infty^2 - [(\gamma-1)/2](\phi_x^2 + \phi_y^2 - \mu u f_x \Delta x - \mu v g_y \Delta y - 1)$$

and

$$\tilde{\rho} = (M_\infty^2 \tilde{a}^2)^{1/(\gamma-1)} \quad (25)$$

Using the binomial expansion and the relations

$$\rho_s \Delta s = -\frac{\rho}{a^2} q q_s \Delta s \approx -\frac{\rho}{a^2}(u q_x \Delta x + v q_y \Delta y) \quad (26)$$

the two variants, Eqs. (25) and (22), are equivalent if f and g are chosen to be $2q$. Notice that Jameson's viscosity form, Eq. (14), is equivalent to the choice of $f=2u$ and $g=2v$ in Eq. (25) as seen from Eq. (15).

III. Discretization Techniques

Standard finite differences are used in discretizing Eq. (11), namely central differencing everywhere, leading to a large system of nonlinear algebraic equations in the unknown potential at the grid points. The finite-difference approximation of Eq. (11) at the grid point (i,j) reads

$$\left\{(\tilde{\rho}_{i+\frac{1}{2},j})\cdot\left(\frac{\phi_{i+1}-\phi_{i,j}}{\Delta x}\right) - (\tilde{\rho}_{i,-\frac{1}{2},j})\cdot\left(\frac{\phi_{i,j}-\phi_{i-1,j}}{\Delta x}\right)\right\}\Big/\Delta x$$
$$+\left\{(\tilde{\rho}_{i,j+\frac{1}{2}})\cdot\left(\frac{\phi_{i,j+1}-\phi_{i,j}}{\Delta y}\right) - (\tilde{\rho}_{i,j-\frac{1}{2}})\right.$$
$$\left.\cdot\left(\frac{\phi_{i,j}-\phi_{i,j-1}}{\Delta y}\right)\right\}\Big/\Delta y = 0 \quad (27)$$

where

$$\tilde{\rho}_{i+\frac{1}{2},j} = \tfrac{1}{2}(\tilde{\rho}_{i+1,j} + \tilde{\rho}_{i,j}), \text{ etc.}$$

and

$$\tilde{\rho}_{i,j} = \tilde{\rho}_{i,j} - \mu_{i,j}(\rho_s \Delta s)_{i,j} \quad (28)$$

$$\rho_{i,j} = (M_\infty^2 a_{i,j}^2)^{1/(\gamma-1)} = M_\infty^2 \left[1 - \frac{\gamma-1}{2}(U_{i,j}^2 + V_{i,j}^2 - 1)\right]^{1/(\gamma-1)} \quad (29)$$

$$U_{i,j} = (\phi_{i+1,j} - \phi_{i-1,j})/2\Delta x \quad (30)$$

$$V_{i,j} = (\phi_{i,j+1} - \phi_{i,j-1})/2\Delta y \quad (31)$$

$$q_{i,j}^2 = U_{i,j}^2 + V_{i,j}^2 \quad (32)$$

$$\mu_{i,j} = \max[0,\ (1 - a_{i,j}^2/q_{i,j}^2)] \quad (33)$$

Upwind differencing is used in evaluating $(\rho_s \Delta s)$ as follows:

$$(\rho_s \Delta s)_{i,j} = \frac{U_{i,j}}{q_{i,j}}\delta_x \rho \Delta x + \frac{V_{i,j}}{q_{i,j}}\delta_y \rho \Delta y \quad (34)$$

where

$$\delta_x \rho \Delta x = \rho_{i,j} - \rho_{i-1,j} \text{ if } U_{i,j} > 0 \quad (35)$$

and

$$\delta_x \rho \Delta x = \rho_{i+1,j} - \rho_{i,j} \text{ if } U_{i,j} < 0 \quad (36)$$

Equations (35) and (36) can be combined in one formula where only central differences are used, i.e.,

$$\delta_x \rho \Delta x = \tfrac{1}{2}(\rho_{i+1,j} - \rho_{i-1,j}) - (E_x/2)(\rho_{i+1,j} - 2\rho_{i,j} + \rho_{i-1,j})$$
$$E_x = \text{sign } U_{i,j} \quad (37)$$

A similar procedure is used to treat the ρ_y term. Together with the discretized boundary condition, the formulation of the discrete problem is complete. Two cases are considered here: 1) flows around a cylinder using cylindrical coordinates with Neumann boundary condition (zero flux) on the cylinder surface and a far-field formula for the outer boundary conditions, and 2) flows around airfoils (parabolic arc airfoil and NACA 0012) with linearized boundary condition at $y=0$ and ϕ is set to be zero in the far field.

In any case, we have a system of equations in the form

$$N\{\phi\} = \{f\} \quad (38)$$

where N is a large, sparse matrix function of $\rho_{i,j}$ (and hence $\phi_{i,j}$). The system of Eq. (38) is solved iteratively as discussed in the next section.

We like to mention here that, with the new formulation, transonic flow problems are amenable for finite-element analysis and computations. In fact, standard Galerkin finite elements can also be used in discretizing Eq. (11) leading to a system of equations similar to Eq. (38). Eberle[7] succeeded to obtain good results with standard bilinear isoparametric elements. Refining the elements leads to a solution which contains embedded shocks and conserves mass. In general, the solution obtained by the artificial compressibility methods seems to depend on a parameter in the expression for the modified density corresponding to the associated artificial viscosity. A physically unacceptable expansion upstream of the shock‡ is sometimes noticed and is eliminated by in-

‡Recently, the second author (South) found that overshoots stem from one source; namely, the way we calculated the density at the nodel points. If a more compact approximation of the density at the midpoints or in the center of the mesh (see Refs. 4 and 5) is used, a sharp shock is obtained with no overshoots, and with no need to augment the viscosity, at least for the cases examined here, which, in fact, bring in values of ϕ from $i-2$ to $i+2$ to calculate ρ at $i+\frac{1}{2}$ by averaging ρ_i and ρ_{i+1}.

creasing the magnitude of the artificial viscosity (a factor of 1.3 is used here).

IV. Iterative Procedures

The convergence of different iterative procedures can be studied through an artificial time-dependent equation describing the development of the solution during iteration. In this section, the application of SOR, ADI, Explicit and Direct Methods are examined.

Line Relaxation Methods

Applying a standard-line relaxation method, marching in the x direction, to the modified density equation in discrete form [Eq. (27)], assuming $\bar{\rho}$ is known from previous iteration, we have

$$\left\{\bar{\rho}^{n}_{i+\frac{1}{2},j}\frac{\phi^{n}_{i+1,j}-\overline{\phi^{n+1}_{i,j}}}{\Delta x}-\bar{\rho}^{n}_{i-\frac{1}{2},j}\frac{\overline{\phi^{n+1}_{i,j}}-\phi^{n+1}_{i-1,j}}{\Delta x}\right\}\Big/\Delta x$$
$$+\left\{\bar{\rho}^{n}_{i,j+\frac{1}{2}}\frac{\overline{\phi^{n+1}_{i,j+1}}-\overline{\phi^{n+1}_{i,j}}}{\Delta y}-\bar{\rho}^{n}_{i,j-\frac{1}{2}}\frac{\overline{\phi^{n+1}_{i,j}}-\overline{\phi^{n+1}_{i,j-1}}}{\Delta y}\right\}\Big/\Delta y=0 \quad (39)$$

and

$$\phi^{n+1}=\phi^{n}+\omega(\overline{\phi^{n+1}}-\phi^{n}) \quad (40)$$

or

$$\overline{\phi^{n+1}}=\frac{\phi^{n+1}-\phi^{n}}{\omega}+\phi^{n}=\frac{\delta\phi}{\omega}+\phi^{n} \quad (41)$$

Thus, Eq. (39) written in matrix form is given by:

$$T\{\delta\phi\}=-\{R(\phi^{n})\} \quad (42)$$

where T is a tridiagonal matrix, $\delta\phi$ is the correction, and $R(\phi^{n})$ is the residual. If we let

$$\delta\phi=\phi_t\Delta t=W\Delta t \quad (43)$$

and

$$\bar{\rho}^{n}_{i-\frac{1}{2}}\delta\phi_{x_{i-\frac{1}{2}}}=\bar{\rho}^{n}\delta\phi_x-\tfrac{1}{2}(\bar{\rho}\delta\phi_x)_x\Delta x \quad (44)$$

Eq. (39) becomes an approximation to:

$$\alpha W_x+\beta(\bar{\rho}W_x)_x+\gamma(\bar{\rho}W_y)_y+\epsilon W=(\bar{\rho}\phi_x)_x+(\bar{\rho}\phi_y)_y \quad (45)$$

where

$$\alpha=\bar{\rho}^{n}(\Delta t/\Delta x),\quad \beta=-(\Delta t/2),\quad \gamma=-(\Delta t/\omega)$$

$$\epsilon=\left(\frac{2}{\omega}-1\right)\bar{\rho}^{n}\frac{\Delta t}{(\Delta x)^2}+\frac{\bar{\rho}^{n}_{x}}{2}\frac{\Delta t}{\Delta x}$$

For simplification, the space derivatives of $\bar{\rho}$ are neglected in the left-hand side of Eq. (45), and, dividing by $\bar{\rho}$, the left-hand side becomes independent of iteration (i.e., T in Eq. (42) becomes the same for each iteration), and, in this case, Eq. (45) reads

$$\alpha' W_x+\beta W_{xx}+\gamma W_{yy}+\epsilon' W=\{(\bar{\rho}\phi_x)_x+(\bar{\rho}\phi_y)_y\}/\bar{\rho} \quad (46)$$

After one complete sweep, the density $\bar{\rho}$ is updated in the whole flowfield and the process is repeated.

It should be mentioned that Jameson's line relaxation method has essentially the same form as Eq. (45) in the subsonic region. In the supersonic region, Jameson's procedure can be viewed as an approximate simulation to the

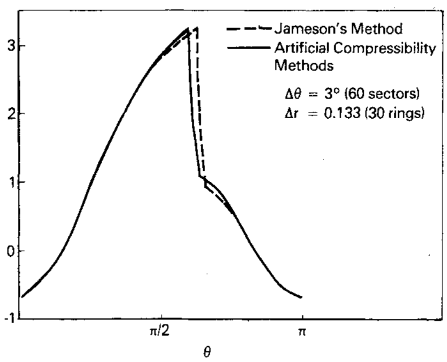

Fig. 1 C_p on surface of cylinder for $M_\infty = 0.51$.

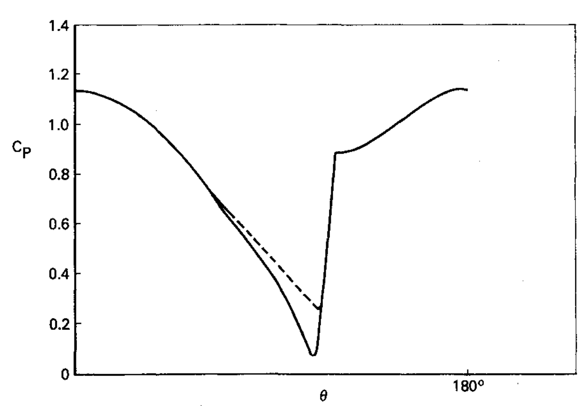

Fig. 2 Isentropic and modified densities on surface of cylinder for $M_\infty = 0.51$.

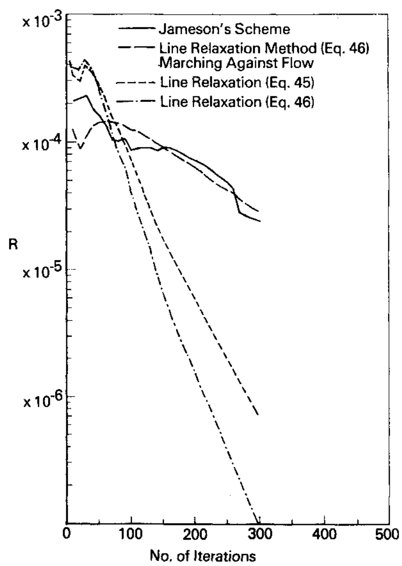

Fig. 3 Convergence of residual for various relaxation schemes.

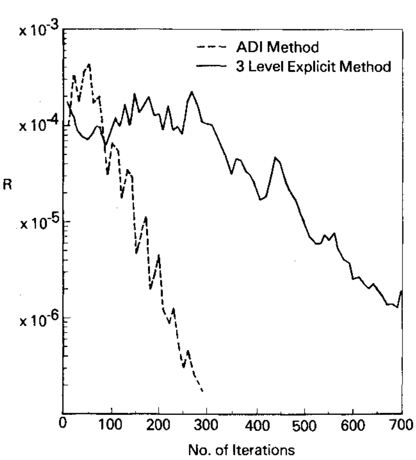

Fig. 4 Convergence of residual for ADI and explicit method.

unsteady low-frequency equation:

$$q\phi_{st} = (a^2 - q^2)\phi_{ss} + a^2\phi_{nn} \tag{47}$$

Based on a Von Neumann analysis, Jameson's scheme is stable. The main difference between Jameson's scheme and the standard line relaxation is due to the ϕ_{st} term in the supersonic region. This term can be implicitly introduced in our scheme by modifying the density with a time-dependent term ϕ_t, namely,

$$\rho = \left(1 - \frac{\gamma - 1}{2} M_\infty^2 (\phi_x^2 + \phi_y^2 + \nu\phi_t - 1)\right)^{1/(\gamma - 1)} \tag{48}$$

Hence,

$$(\rho\phi_x)_x + (\rho\phi_y)_y = (-\rho/a^2)(\nu U\phi_{xt} + \nu V\phi_{yt} - (a^2 - U^2)\phi_{xx} + 2UV\phi_{xy} - (a^2 - V^2)\phi_{yy})$$
$$= (-\rho/a^2)(\nu q\phi_{st} - (a^2 - q^2)\phi_{ss} - a^2\phi_{nn}) \tag{49}$$

The coefficient ν is chosen to be proportional to the switching function μ. With this modification, the new method is very similar to Jameson's except it is simpler to implement. In fact, the ϕ_{st} term is always necessary for convergence, in particular if an overrelaxation parameter close to two is used to produce a fast rate of convergence.

Numerical results of the artificial compressibility method for flows around a cylinder is shown in Fig. 1 and is compared to our version of Jameson's calculations for the same mesh. In Fig. 2, both the isentropic and modified densities on the surface of the cylinder are plotted. Figure 3 shows the history of convergence for Jameson's iterative procedure in terms of the maximum residual R vs number of iterations. The histories of the iterative procedures described by Eqs. (45) and (46) are also shown in Fig. 3. Calculations based on line relaxation marching against the flow direction have been done and the convergence results are presented in Fig. 3. It seems that the iterative procedure described by Eq. (46) converges independent of the sweeping direction, a property which is useful for three-dimensional calculations.

ADI Methods

It has been demonstrated by Ballhaus et al.[10] that modified ADI methods applied to a transonic small-disturbance equation can be faster than line relaxations. Extensions of their work to a full-potential equation is under way.[11] Here we applied the standard ADI method, without any modifications whatsoever, to the full-potential equation in conservative form, with artificial density. The method may be described by

$$(\alpha\tilde{\rho} - \partial_x\tilde{\rho}\partial_x)(\alpha\tilde{\rho} - \partial_y\tilde{\rho}\partial_y)\delta\phi = \alpha\sigma\tilde{\rho}[(\tilde{\rho}\phi_x)_x + (\tilde{\rho}\phi_y)_y] \tag{50}$$

The solution procedure consists of two steps; in each step, only a one-dimensional operator is employed, namely,

$$(\alpha\tilde{\rho}^n - \partial_y\tilde{\rho}^n\partial_y)f = \alpha\sigma\tilde{\rho}R(\phi^n) \tag{51}$$

followed by

$$(\alpha\tilde{\rho}^n - \partial_x\tilde{\rho}^n\partial_x)\delta\phi = f \tag{52}$$

where α and σ are acceleration parameters. It should be mentioned that the convergence rate can be optimized through the choice of α in a standard manner.[10] Without optimizing these acceleration parameters, ADI methods are not superior to LSOR. Results of flows around a cylinder, for the same case as described previously, are shown in Fig. 4.

An even simpler ADI scheme is obtained by dividing Eq. (50) by $\tilde{\rho}^2$ to obtain a stationary iteration scheme i.e., the iteration matrix does not change from one iteration to the next) as follows, ($\sigma = 2$):

$$(\alpha - \partial_{xx})(\alpha - \partial_{yy})\delta\phi = (\alpha\sigma/\tilde{\rho})[(\tilde{\rho}\phi_x)_x + (\tilde{\rho}\phi_y)_y] \tag{53}$$

This simple scheme has been tested successfully.§ In difficult transonic cases where there is a large region of supersonic flow and strong shocks, terms $\beta_1\partial_x$ and $\beta_2\partial_y$ are added to the first and second factors of the left hand side of Eq. (53), respectively. This addition produces ϕ_{xt} and ϕ_{yt} terms in the associated artificial time-dependent equation describing the development of the solution through iteration. Equation (53) thus becomes

$$(\alpha + \beta_1\partial_x - \partial_{xx})(\alpha + \beta_2\partial_y - \partial_{yy})\delta\phi = (\alpha\sigma/\tilde{\rho})R(\phi^n) \tag{54}$$

§In fact, the stationary scheme, even for stretched meshes, shows no loss of convergence rate; hence, it is more efficient.

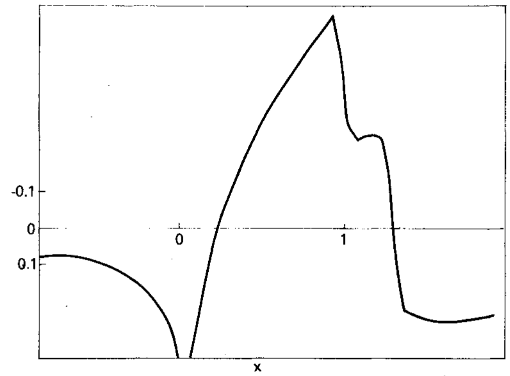

Fig. 5 C_p vs x parabolic arc airfoil 10%, $M_\infty = 0.98$—three-level explicit method (ACM).

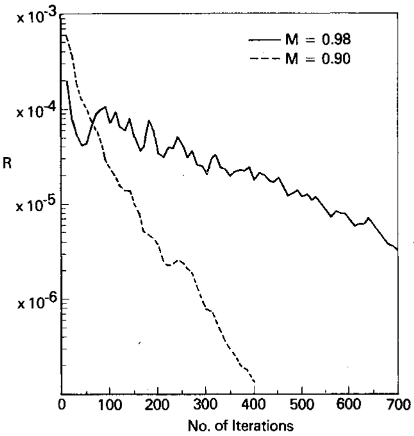

Fig. 6 Convergence of residual for parabolic arc calculations using explicit method.

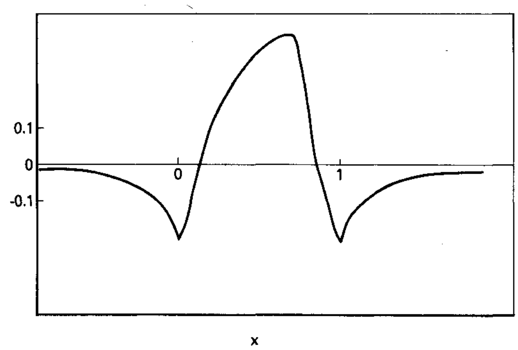

Fig. 7 C_p vs x parabolic arc airfoil 6%, $M_\infty = 0.9$—three-level explicit method (ACM).

Upwind differences are used for the $\beta_1\partial_x$ and $\beta_2\partial_y$ terms. The effect of $\beta_2\partial_y$ has not been tested yet in a general case.

Explicit Methods

Although explicit methods are generally slow, recently there is a revived interest in such methods which can be easily vectorized. Keller and Jameson[12] solved the transonic small-disturbance equation in nonconservative form using a three-level iterative procedure on the CDC STAR 100 computer, which is a vector array processor. We applied standard explicit methods on the modified density equation. First, the two-level explicit method:

$$\bar\rho\delta\phi=\sigma[(\bar\rho\phi_x)_x+(\bar\rho\phi_y)_y] \tag{55}$$

was extremely slow, as should be expected; therefore, it was abandoned. A three-level method, (representing a damped wave equation) described by:

$$\alpha\bar\rho\delta^2\phi+\beta\bar\rho\delta\phi=(\bar\rho\phi_x)_x+(\bar\rho\phi_y)_y \tag{56}$$

where α and β are chosen to optimize the rate of convergence was tested. Convergence results for flow around a cylinder are shown in Fig. 4. Computed pressures and convergence histories for flow around a parabolic arc airfoil with linearized boundary conditions are shown in Figs. 5 and 6 for $M_\infty=0.98$ and in Figs. 6 and 7 for $M_\infty=0.9$. In Fig. 8, the iterative history of a horizontal line relaxation procedure and the three-level explicit scheme for calculation of a flow around NACA 0012 ($M_\infty=0.85$) are presented.

Unlike some line relaxation methods, no difficulties are encountered for high freestream Mach numbers, including supersonic flows, in the case of the explicit method. The rate of convergence is slower for the higher Mach numbers as expected and can be improved by explicitly adding a ϕ_{st} term to the scheme as discussed before.

Direct Methods

The following implicit-iterative procedure (usually used in finite-element calculations of subsonic flows) has been tested:

$$(\tilde\rho^n\phi_x^{n+1})_x+(\tilde\rho^n\phi_y^{n+1})_y=0$$

or

$$(\tilde\rho^n\delta\phi_x)_x+(\tilde\rho^n\delta\phi_y)_y=-(\tilde\rho^n\phi_x^n)_x-(\tilde\rho^n\phi_y^n)_y \tag{57}$$

A full matrix is inverted at each iteration. The method diverges, calling for fuller treatment of the nonlinearity (e.g., Newton's method). In order to obtain a convergent procedure, a damping term proportional to $\delta\phi$ is added to the left-hand side of Eq. (57) and the right-hand side is multiplied by an underrelaxation factor [i.e., we have an implicit version of Eq. (55)]. The convergence rate is painfully slow. The same conclusion holds for a similar, but simpler procedure, where a Laplacian L is used to replace the operator in the left-hand side of Eq. (57), namely,

$$\bar\rho(L\delta\phi+\omega\delta\phi)=-\sigma R(\phi^n) \tag{58}$$

where a fast solver may be used at each iteration. Implicit versions of three-level methods based on Eq. (56) are expected to have a faster convergence rate but will not be discussed here.

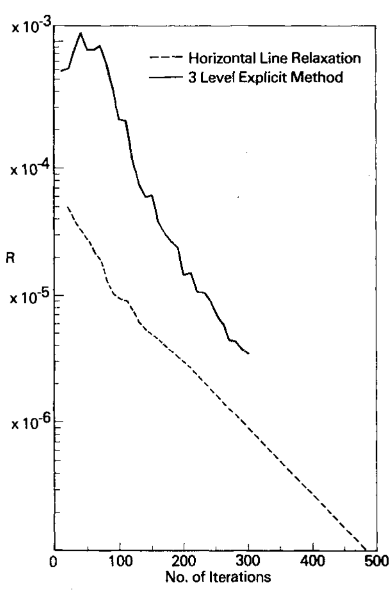

Fig. 8 Convergence of residual for NACA 0012 airfoil, $M=0.85$.

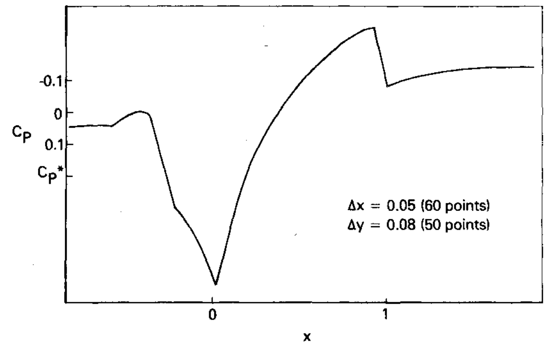

Fig. 9 C_p on surface of 5% paabolic arc airfoil at $M_\infty=1.1$—explicit three-level scheme on STAR 100 computer.

V. Vector Processing and the STAR 100 Computer

It should be emphasized that the difference scheme and iterative methods presented in this paper are not necessarily faster methods for treating transonic flows; they are indeed simpler and more compact, however, than previous methods. The simplicity and the demonstrated amenability to solution by many elliptic-type iterative schemes (including stationary ones), offer an approach to certain problems that were in-

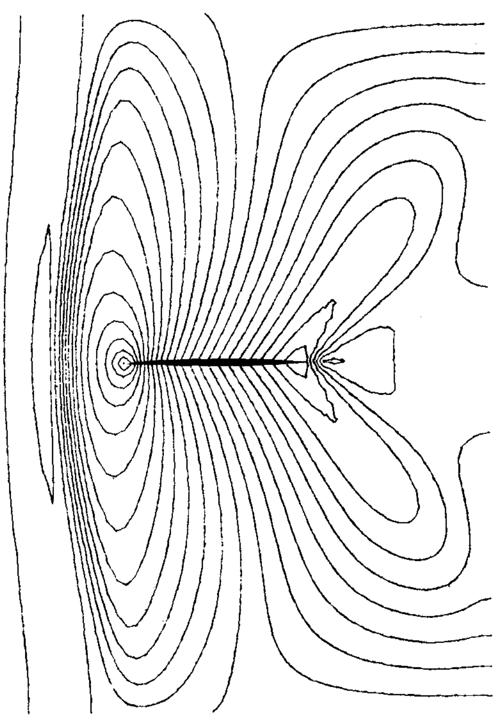

Fig. 10 Contours of constant Mach number for supersonic freestream flow ($M_\infty=1.1$, 5%-thick parabolic-arc airfoil).

tractable by other methods. An example of this feature is the application of the artificial compressibility method coupled with the three-level explicit iteration to vector processing.

An elegant scheme for such purposes was reported by Keller and Jameson[12] for the two-dimensional transonic small-disturbance equation in nonconservative form. Calculations were carried out on the CDC STAR 100 vector processor at NASA Langley Research Center, demonstrating that significant gains in computer speed could be made by using somewhat slower, but easily vectorizable, explicit methods, rather than the existing semi-implicit methods such as SLOR. It was not known, however, how to extend their method to the full potential conservative form.

The three-level explicit scheme already discussed herein was easily programmed on STAR (by James D. Keller, Langley Research Center), and the speed gains were found to be surprisingly impressive. Both STAR FORTRAN[13] and the newer SL/1 language[14] were used to code the unstretched grid problem. The SL/1 language allows the use of the half-word (32 bits) capability of STAR, which in turn yields about twice the speed and storage of the full-word operations.

In order to provide a still more severe test of the present method, a supersonic freestream case, $M_\infty = 1.1$, is presented. This case is for a 5% thick parabolic arc airfoil and a 60×50 mesh, with the disturbance potential set to zero on the front, top, and bottom boundaries and extrapolation at the back boundary. There is a bow shock in front of the airfoil and an oblique shock at the trailing edge, as shown in Figs. 9 and 10. Convergence is, of course, quite slow, but most other relaxation methods would diverge in this case. The STAR run was allowed to continue for 10,000 cycles to assure convergence, since a good engineering criteria for convergence was not available. The 10,000-cycle run required only 45 s. Examination of the output indicated that accurate engineering results were obtained after about 3000 cycles, which required only 14 s. The computation rate with the SL/1 code was 637,462 points/s. In contrast, the computation rate on the CYBER 175 for the same method was 23,800 points/s. Finally, the computer time required on the CYBER 175 to obtain similar engineering accuracy, using our ADI method, Eq. (54), was 38 s (300 cycles), and the computational rate was 22,250 points/s.

In all the numerical results presented, the finite-difference meshes used were quite coarse as well as equally spaced. We would like to mention that we have recently used fine stretched meshes without any trouble. Of course, the convergence rate deteriorates slightly as expected, for smoothly stretched Cartesian meshes. However, the present methods have been found more reliable than any existing method that the authors have experienced.

VI. Conclusions

We presented a new method for the numerical solution of full-potential equation, in conservation form, based on modifying the density with space as well as time derivative terms. The former introduces an effective artificial viscosity, while the latter affects the artificial time-dependent equation describing the particular iterative procedure used in the solution of the discrete problem. It is shown that standard discretization techniques (central differencing and also finite elements) and some standard iterative procedures (SOR, ADI, and explicit methods) are applicable to the compact form of the modified density equation. The nonlinear mixed-type equation is treated as if it were an elliptic equation, where the modified density is assumed to be known from the previous iteration. Standard acceleration techniques are effective (for example, optimized ADI, extrapolation[15] and multigrid methods[16]). The explicit method, in particular, is suitable for vector processing. A feasibility study on the STAR 100 computer demonstrates the advantage of the new method due to its simplicity. Extensions to three-dimensional calculations are straightforward.

Acknowledgment

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